[The following Smith's Prize Exam was taken by James Clerk Maxwell at Cambridge. Question 8 is Stokes' Theorem. (Stokes was a personal friend of Maxwell.) Maxwell completed the exam tied for first.]
[Transcription errors in questions 1, 3, 4, and 15 corrected and a note added to question 7, 04 April 2005 -- JCR]

February, 1854.

BY GEORGE GABRIEL STOKES, ESQ. M. A.

Lucasian Professor.

1. STRAIGHT lines AP, BP pass through the fixed points A, B, and are always equally inclined to a fixed line; shew that the locus of P is a hyperbola, and find its asymptotes.

2. A number of equal vessels communicate successively with each other by small pipes, the last vessel opening into the air. The vessels being at first filled with air, a gas is gently forced at a uniform rate into the first; find the quantity of air remaining in the n^{th} vessel at the end of a given time, supposing the gas and air in each vessel at a given instant to be uniformly mixed.

3. Separate the roots of the equation

$$2x^3 - 9x^2 + 12x - 4.4 = 0,$$

and find the middle root to four places of decimals by Horner's method, or by some other.

4. Investigate a formula in Finite Differences for transforming a series the terms of which (at least after a certain number) are alternately positive and negative, and decrease slowly, into one which is generally much more rapidly convergent.

Example. Find the sum of the series

1.4142 – .7071 + .5303 – .4419 + .3867 – .3480 + .3190 – .2962 + …

5. Given the centre and two points of an ellipse, and the length of the major axis, find its direction by a geometrical construction.

6. Integrate the differential equation

$$(a^2 - x^2)dy^2 + 2xydydx + (a^2 - y^2)dx^2 = 0$$

Has it a singular solution?

7. In a double system of curves of double curvature, a tangent is always drawn at the variable point P; shew that, as P moves away from an arbitrary fixed point Q, it must begin to move along a generating line of an elliptic cone having Q for vertex in order that consecutive tangents may ultimately intersect, but that the conditions of the problem may be impossible.

8. If X, Y, Z be functions of the rectangular co-ordinates x, y, z, dS an element of any limited surface, l, m, n the cosines of the inclinations of the normal at dS to the axes, ds an element of the bounding line, shew that

$$\iint\left\{l\left(\frac{dZ}{dy}-\frac{dY}{dx}\right)+m\left(\frac{dX}{dz}-\frac{dZ}{dx}\right)+n\left(\frac{dY}{dx}-\frac{dX}{dy}\right)\right\}dS$$
$$=\int\left(X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}\right)ds,$$

the differential coefficients of X, Y, Z being partial, and the single integral being taken all round the perimeter of the surface.

9. Explain the geometrical relation between the curves, referred to the rectangular co-ordinates x, y, z, whose differential equations are

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R},$$

and the family of surfaces represented by the partial differential equation

$$P\frac{dz}{dx}+Q\frac{dz}{dy}=R.$$

10. Write a short dissertation on the theoretical measure of mass. By what experiments did Newton prove that masses may be measured by their weights? Independently of such experiments, how may it be inferred from the observed motions of the heavenly bodies that the mutual gravitation of two bodies depends only on their masses, and not on their nature? In what two different senses is the term *weight* used?

11. What are the conditions to be satisfied in order that two moving systems may be dynamically as well as geometrically similar?

If it be desired to investigate the resistance to a canal boat moving 6, 8, 10 miles an hour by experiments made with a small model of the boat and canal, if the boat be 36 feet long and its model only 2 ft. 3 in., what velocities must be given to the latter.

12. A rod is suspended at two given points by unequal light elastic strings, in such a manner that the rod is horizontal and the strings are vertical in the position of equilibrium; the rod being slightly disturbed in a vertical plane, in such a manner that no

displacement of velocity is communicated to the centre of gravity in a horizontal direction, it is required to determine the motion.

13. Shew how to determine the time of rotation of the Sun about his own axis, and the position of his equator.

14. Rays coming from a luminous point situated in the axis of a large convex lens, and beyond the principal focus, are received after transmission through the lens on a screen field perpendicular to the axis, which is moved from a little beyond the extremity of the caustic surface to a little beyond the geometrical focus; compare, according to the principles of geometrical optics, the illumination at different points of the screen, and at different points of the screen, and at different distances of the screen from the lens, the lengths of any lines in the figure being regarded as known.

Give a sketch of the method of finding the illumination in the neighbourhood of a caustic according to the theory of undulations. What is the general character of the result, and in what natural phenomenon is it exhibited?

15. A glass plate, the surface of which is wetted, is placed vertically in water; shew that the elevation of the fluid varies as the sine of half the inclination of the surface to the horizon, and compare its greatest value with the elevation in a capillary tube of given diameter. Find also the equation of the surface.

16. Explain the different modes of determining the Mass of the Moon.

17. Plane polarized light is transmitted, in a direction parallel to the axis of the crystal, across a thick plate of quartz cut perpendicular to the axis, and the emergent light, limited by a screen with a slit, is analyzed by a Nicol's prism combined with an ordinary prism; describe the appearance presented as the Nicol's prism is turned round, and from the phenomena deduce the nature of the action of quartz on polarized light propagated in the direction of the axis.

[Following Smith's Prize Exam was written by James Clerk Maxwell. Maxwell died in November of this year.]

WEDNESDAY, *January* 29, 1879

BY J. CLERK MAXWELL, PROFESSOR OF EXPERIMENTAL PHYSICS.

1. IF in a plane $AB = CD$, and if P be the intersection of the lines which bisect AC and BD perpendicularly, Q that of the lines which bisect AD and BC perpendicularly, and R the intersection of AB and CD, shew that PR and QR are the interior and exterior bisectors of the angle ARC.

2. Three points A, B, C on a straight line correspond homographically to three points a, b, c on another straight line: give a geometrical construction to determine the point on either line which corresponds to a point at an infinite distance on the other.

Four points A, B, C, D, of which no three are in a straight line, correspond homographically to four points a, b, c, d: prove the following construction for finding the equiangular foci of the two figures, at which corresponding lines subtend equal angles.—Let Y, Z, y, z be the vanishing points on the lines AC, AB, ac, ab respectively; on YZ describe the triangle ZF_1Y similar to zay, and on yz describe the triangle zf_1y similar to ZAY, then F_1 and f_1 are the positive equiangular foci, and their images in YZ and yz respectively are the negative foci.

3. Shew that if two points are taken at random in a region of volume V, the probability of their distance being between a and $a+da$, where a is small compared with the dimensions of the region, is

$$4\pi V^{-1}a^2 da.$$

Shew also that if three and four points are taken at random, the probabilities of their being within specified limits of distance are

$$8\pi^2 V^{-2} abcdadbdc \text{ and } \frac{8}{3}\pi^2 V^{-3} T^{-1} abclmndadbdcdldmdn$$

respectively, where a, b, c, l, m, n are the distances between the points, and T is the volume of the tetrahedron whose angles are the four points.

4. Thirty rods of equal length have their ends jointed five and five together so as to form the edges of a regular icosahedron: an elastic string is stretched between two opposite angles of the figure. Shew that if the tension of the string is $2\sqrt{5}\left(10-2\sqrt{5}\right)^{\frac{1}{2}}$, the ten rods which meet the string will each have a pressure 4; the ten rods which are

perpendicular to the string will each have a tension $\sqrt{5}+1$, and the other 10 rods will each have a pressure $\sqrt{5}-1$.

5. Three grooves are cut in a horizontal table; the bottoms of the grooves are horizontal lines meeting in a point O at angles of 120°; the sides of the grooves are planes inclined 45° to the vertical: a tripod is placed with its feet in the grooves, each foot being distant a from the point O. Suppose one of the feet is due east of O, and that it is made to slide up the south side of the groove in which it stands: shew that the tripod will begin to move about a right hand screw, whose pitch is $\frac{3}{5}a$, and whose axis meets the plane of the feet at a point $\frac{4}{5}a$ due west of the point O, and is inclined $\tan^{-1}2$ in the plane of the meridian, measured from the zenith towards the north.

6. If the centres of the bodies of the solar system are projected perpendicularly on a fixed plane, and the forces which act on them are resolved in that plane, shew that the second differential coefficient with respect to the time of the moment of inertia of the system about an axis through its centre of mass perpendicular to that plane, together with twice the sum of the products of the projection of the distance between each pair of bodies into the resolved part of the attraction between them, is equal to four times the kinetic energy due to that part of the motion of the system relative to its centre of mass which is parallel to the plane of reference.

7. A thin, uniformly elastic rod, $OABC$, originally straight, is constrained to pass through the points A, B, C in a straight line. Shew that the deflection of the part OA, produced by forces acting on that part only, will be the same as if the rod had been constrained to pass through two points A and X only, where

$$AX = AB\frac{3AB+4BC}{4AC}.$$

If the rod is constrained to pass through an infinite number of points, at intervals each equal to AB, shew that the constraint, as regards the point[1] OA, will be the same as if the rod had been constrained to pass through A and Y, where

$$AY = \frac{1}{2}\sqrt{3}AB.$$

8. The configuration of four particles, whose masses are P, Q, R, S, is determined by their distances $QR = a$, $RP = b$, $PQ=c$, $PS=l$, $QS=m$, $RS=n$, and the potential energy of the system is

$V = \frac{1}{2}\left[A(a-a_0)^2 + B(b-b_0)^2 + C(c-c_0)^2 + L(l-l_0)^2 + M(m-m_0)^2 + N(n-n_0)^2\right]$:

shew that the small oscillations of the system are determined by six equations of the form

[1] This appears to be an error in the original text; it should read "part OA", thanks to D. Forfar for this observation.

$$QRa + (Q+R)A(a-a_0) + QB(b-b_0)\cos ab + RC(c-c_0)\cos ac$$
$$+ RM(m-m_0)\cos am + QN(n-n_0)\cos an = 0$$

where cos *ab* denotes the cosine of the angle *QRP* between *a* and *b*.

Shew also that if the particles are all equal, and the law of force such that any two of them would be in equilibrium at a distance *a* and would make small oscillations of period *T*, then for three such particles the periods of the fundamental vibrations are $\sqrt{\tfrac{4}{3}}\,T$ and $\sqrt{\tfrac{2}{3}}\,T$, and for four such particles $\sqrt{2}\,T$, T, and $\tfrac{1}{2}\sqrt{2}\,T$.

9. Define the principal foci and the principal focal length of an optical instrument, and shew that in any system of thin lenses having the same axis:–

(1) The reciprocal of the principal focal length is the sum of the reciprocals of the focal lengths of the lenses, together with the sum of all intervals and products of consecutive intervals into which the axis may be divided by the lenses, each product being divided by the product of the focal lengths of the lenses at the points of section, including the first and last.

(2) The distance of the first principal focus of the instrument from the first lens is equal to the principal focal length of the system multiplied by 1 + the sum of all intervals and products of consecutive intervals beginning with the first lens, each divided by the product of the focal lengths of the lenses at the points of section excluding the first lens.

In what direction is this distance to be measured?

10. The motion of an incompressible homogeneous fluid in a spherical vessel at a given instant is such that each spherical stratum rotates like a rigid shell, the rectangular components of its angular velocity being ω_1, ω_2, ω_3, these quantities varying from stratum to stratum: shew that if each particle is attracted towards the centre with a force whose intensity per unit of mass is

$$(\varpi_1 x + \varpi_2 y + \varpi_3 z)\left(x\frac{d\varpi_1}{dr} + y\frac{d\varpi_2}{dr} + z\frac{d\varpi_3}{dr}\right) + \frac{dV}{dr},$$

where *V* is any function of the co-ordinates, the motion of the fluid will be steady, and determine the pressure at any point.

11. Shew that if *P* be a variable point on a sphere, and *A*, *B*, fixed points,

$$3\cos PA\cos PB - \cos AB$$

is a spherical harmonic of the second order, and that if *A*′, *B*′, are the poles of another harmonic, the condition of the two harmonics being conjugate to each other is

$$3\cos AA'\cos BB' + 3\cos AB'\cos A'B - 2\cos AB\cos A'B' = 0$$

If the intersections of five equidistant meridians with two parallels of latitude are the poles of five spherical harmonics of the second order, shew that if they are all conjugate to each other, the polar distances of the two circles must be

$$\tan^{-1}\left(2+\sqrt{6}\pm\sqrt{14+6\sqrt{6}}\right).$$

12. A soap bubble is gradually charged with electricity; determine the pressure of the air within it, and shew that when it becomes less than that of the air outside, the equilibrium of the bubble becomes unstable with respect to small deviations from the spherical form.

13. The resistance of a battery with its electrodes is R, and its electromotive force is constant. The circuit is completed by a fine wire of uniform section, whose resistance is a function of its temperature, which is supposed to be the same at all points of the wire. The wire is such that if no heat were generated in it, it would lose one per cent. of its excess of temperature over that of the air in a time T; and the electromotive force is such that if the wire were prevented from losing heat, its resistance would increase one per cent. in time τ. Shew that if r is the resistance of the wire when the current is in equilibrium, the equilibrium will be unstable if $(R-r)T$ is greater than $(R+r)\,\tau$.

14. It is proposed to construct a resistance coil the percentage error of which shall be a minimum: the probable error arising from imperfect connexion of the electrodes is r, and the defect of insulation is such that, independently of the wire, the conductivity between the electrodes is C, with a probable error e: shew that the best value for the resistance of the wire is such that if x is the actual resistance of the apparatus,

$$x^4c^2=(1-Cx)^3(1+Cx)r^2.$$

15. The measured values of the sides of a certain triangle are a, b, c, and the observed angles are found to exceed the angles calculated from a, b and c by X, Y and Z respectively: shew that the most probable values of the sides are $a(1+x)$, $b(1+y)$ and $c(1+z)$, where x, y, z are given by the equations

$$\left[3a^4+(b^2-c^2)^2+8\Delta^2\frac{a^2}{l^2}\right]x+\left[c^4-\left(a^2+b^2\right)^2+2\Delta^2\right]y$$
$$+\left[b^4-\left(c^2+a^2\right)^2+2\Delta^2\right]z=2\Delta\left[2a^2X+(c^2-a^2-b^2)Y+(b^2-c^2-a^2)Z\right]$$

$$\left[c^4-\left(a^2+b^2\right)^2+2\Delta^2\right]x+\left[3b^4+(c^2-a^2)^2+8\Delta^2\frac{b^2}{l^2}\right]y$$
$$+\left[a^4-\left(b^2+c^2\right)^2+2\Delta^2\right]z=2\Delta\left[\left(c^2-a^2-b^2\right)X+2b^2Y+(a^2-b^2-c^2)Z\right],$$

$$\left[b^4 - \left(c^2 + a^2\right)^2 + 2\Delta^2\right]x + \left[a^4 - \left(b^2 + c^2\right)^2 + 2\Delta^2\right]y$$
$$+\left[3c^4 + \left(a^2 - b^2\right)^2 + 8\Delta^2 \frac{c^2}{l^2}\right]z = 2\Delta\left[\left(b^2 - a^2 - c^2\right)X + \left(a^2 - b^2 - c^2\right)Y + 2c^2 Z\right],$$

where Δ is the area of the triangle, and the probable error of a measurement of length is supposed to be l times that of a measurement of angle.