A Model for Discretization Error in Electromagnetic Analysis of Capacitors

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***Abstract*— The error due to discretization in a method-of-moments analysis of a parallel plate or metal–insulator–metal (MIM) capacitor is discussed. A technique related to Richardson extrapolation is used to develop a model for the error due to subsectional discretization. The results are for Galerkin's method using rooftop basis functions; however, the technique can be applied to any variational moment-method calculation. An expression is presented for the error in capacitance calculations, which is shown to hold for changes in geometry and dielectric constant. In addition, the expression for error is shown to be accurate for a wide range of meshing geometries. Surprisingly, the error model is not an upper bound, but rather is met nearly in equality for all geometries considered. Thus, the error may be simply subtracted from the calculated value for a more accurate result.**

***Index Terms*—Capacitance calculation, discretization error.**

I. Introduction

Galerkin method-of-moments analyses exhibit a variational property, i.e., as the number of basis functions used approaches infinity, the numerical solution converges to the exact solution [1] to the extent allowed by the numerical precision used. This principle has been applied to the error analysis of a stripline transmission line used as a benchmark since its exact solution is known [2]. For the electromagnetic analysis of parallel plate or MIM (metal–insulator–metal) capacitors, a similar analysis is performed here by isolating the discretization error in each direction and observing the convergence behavior.

The key to the proposed method is to consider the final answer (the capacitance) of the calculation to be a function of the discretization level or number of cells in a given direction M, and then to calculate the capacitance for several values of M. A function can be fitted to the results of these calculations which is in turn evaluated at the desired (very time consuming) discretization to extrapolate capacitance values with higher accuracy. This technique is generally used in Romberg integration, and is known as Richardson extrapolation [8], [9].

For example, a square capacitor in the X–Y plane can be first discretized with a very small (high resolution) cell size in the X-direction while the number of cells in the Y-direction is varied. This is then repeated in the X-direction with a fixed fine meshing in the Y-direction. Using a spreadsheet, the convergence trend of the individual error sources (X, Y discretization) can be observed and an error model fitted to the data. This model is then evaluated for a cell size of zero.

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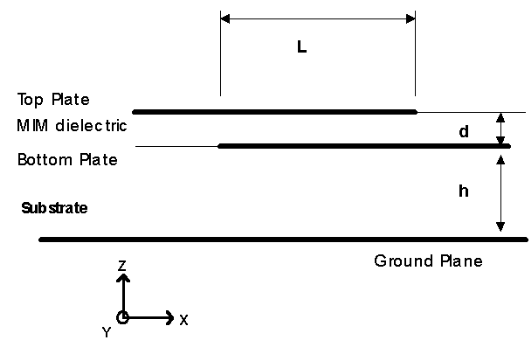

Fig. 1. MIM series capacitor cross-section layout used in the example.

II. Discussion

As a practical example of the procedure, a MIM capacitor (shown in Figs. 1 and 2) 0.5-mm square with dielectric constant 10.0 and dielectric thickness 100 nM is modeled on a 10-mm-thick substrate 1-mm square using Sonnet ***em***.[1] A full description of this software is given in [3]. ***em***'s Spice option [4] determines series capacitance and port discontinuity capacitance separately, so de-embedding of the port discontinuity is not needed.

The capacitor was discretized with 256 cells across its length (X-direction) and the discretization in the width (Y-direction) is varied from 2 to 256 cells. This is then repeated holding the Y-directed discretization at 256 and varying the X-direction cell count. A summary of the results is shown in Table I.

The analyses were performed at 1.0 MHz to make sure parasitic inductances and capacitances were not confused with the desired capacitance error. Although not the subject of this paper, error models for the parasitics can also be determined. At higher frequencies, effects due to the planar structure and the ground plane also become significant. For example, an equivalent circuit representation [5] for a MIM series capacitor is shown in Fig. 3.

At low frequencies, C_{MIM} dominates and is what is considered in this paper. C_{MIM} is the sum of the parallel-plate

[1] Sonnet Software Inc., Liverpool NY.

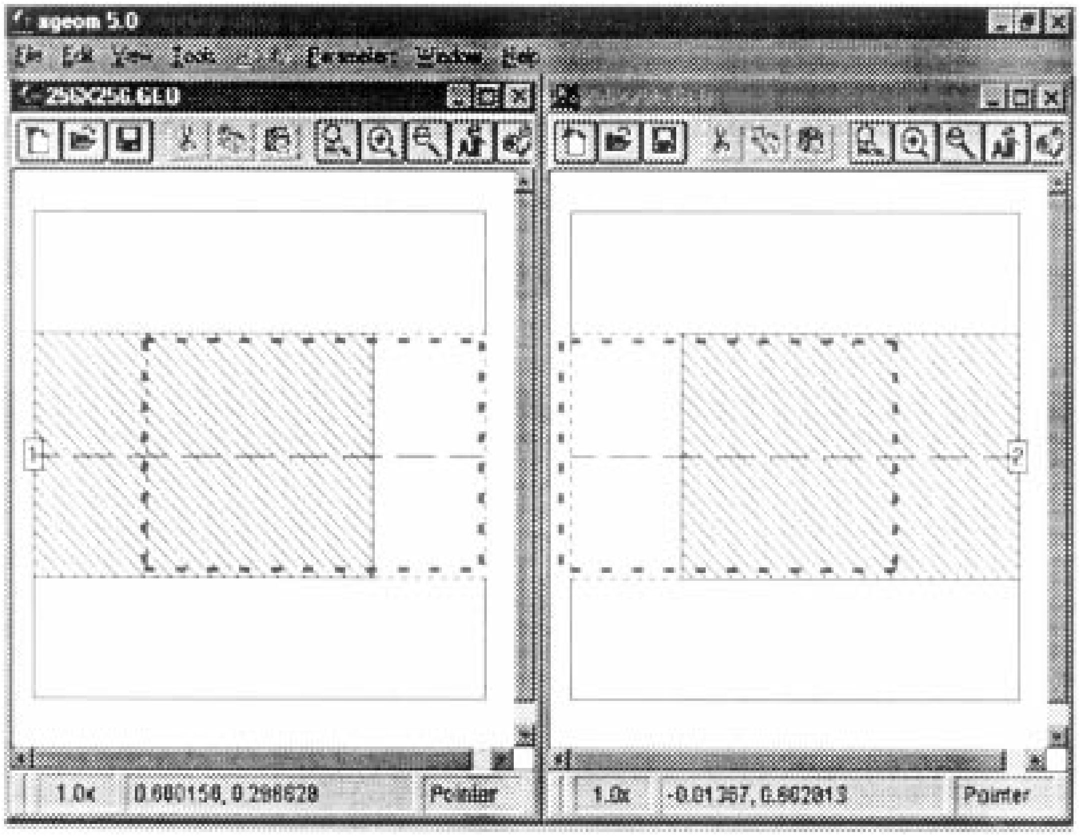

Fig. 2. Two top views (x, y plane) of the Sonnet *xgeom* description of the MIM series capacitor. The left view shows the top plate (cross hatch) with the bottom plate in dotted line and the right view shows the bottom plate with top plate in dotted line. The two plates overlap up to the dotted lines.

TABLE I
THE BASELINE CONVERGENCE ANALYSIS
RESULTS FOR A 0.5-MM-SQUARE CAPACITOR

Cells/Length M	Cells/Width N	Cap (pF)	Change (pF)	Change (%)
256	2	227.25513	—	—
256	4	224.38529	−2.870	−1.279%
256	8	222.94062	−1.445	−0.648%
256	16	222.23118	−0.709	−0.319%
256	32	221.87358	−0.358	−0.161%
256	64	221.69608	−0.178	−0.080%
256	128	221.60892	−0.087	−0.039%
256	256	221.56736	−0.042	−0.019%
2	256	227.20116	—	—
4	256	224.33293	−2.868	−1.279%
8	256	222.90571	−1.427	−0.640%
16	256	222.19704	−0.709	−0.319%
32	256	221.84819	−0.349	−0.157%
64	256	221.67935	−0.169	−0.076%
128	256	221.60070	−0.079	−0.035%
256	256	221.56736	−0.033	−0.015%

capacitance and the very small, but potentially important, fringing capacitance around the edge of the capacitor plates. This technique can also be applied to compute the bottom plate to ground capacitance C_p if desired. For planar shunt capacitors, the same results apply and can be used to model $C_{\rm sh}$ in Fig. 4 from [6].

Note in Table I that the change is reduced by half when doubling the number of cells in either the X- or Y-direction.

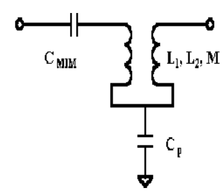

Fig. 3. Equivalent circuit model of a series MIM capacitor from [5].

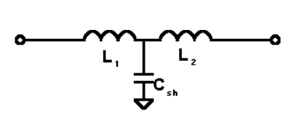

Fig. 4. Equivalent circuit model of a shunt MIM capacitor from [6].

The small differences between the x and y cases are due to the feedline positions, since in the first case $(256 \times N)$ the feedline has a different discretization than the second case $(M \times 256)$. Looking at the convergence data in each case, we can conclude that the total remaining error at a given discretization level is very nearly equal to the change from one level to the next. If this pattern continues for $M, N \geq 256$, we can assume the total error left in the 256 $\times$ 256 cell result is 0.042 pF $+$ 0.033 pF. If true, then the converged result is 221.567 pF $-$ 0.042 pF $-$ 0.033 pF $=$ 221.492 pF with a subjectively estimated error of ±0.01 pF, or ±0.0045%. Note that at the other extreme, i.e., without using convergence

TABLE II
CONVERGENCE ANALYSIS RESULTS FOR THE SAME CAPACITOR, BUT WITH DIELECTRIC THICKNESS 200 nM. NOTE THE SIMILARITY WITH THE BASELINE CASE IN TABLE I

Cells/Length M	Cells/Width N	Cap (pF)	Change (pF)	Change (%)
256	2	113.68227	—	—
256	4	112.23747	−1.435	−1.278%
256	8	111.52983	−0.718	−0.643%
256	16	111.17211	−0.358	−0.322%
256	32	110.99456	−0.178	−0.160%
256	64	110.90738	−0.087	−0.079%
256	128	110.86583	−0.042	−0.037%
256	256	110.84756	−0.018	−0.016%
2	256	113.62954	—	—
4	256	112.19228	−1.437	−1.281%
8	256	111.48803	−0.704	−0.632%
16	256	111.13903	−0.349	−0.314%
32	256	110.97014	−0.169	−0.152%
64	256	110.89145	−0.079	−0.071%
128	256	110.85811	−0.033	−0.030%
256	256	110.84756	−0.011	−0.010%

TABLE III
CONVERGENCE ANALYSIS RESULTS WITH THE BASELINE CONFIGURATION INSULATOR DIELECTRIC CONSTANT CHANGED TO 1.0. EXCEPT FOR THE VERY LOW ERROR RANGE. THE RESULTS ARE STILL SIMILAR TO THE BASELINE RESULTS. THE LOW ERROR DISCREPANCIES ARE CAUSED BY NUMERICAL PRECISION ERROR DUE TO THE LOW ANALYSIS FREQUENCY. THIS SITUATION ARISES WHEN SUBSECTIONS ARE LESS THAN 0.00001 WAVELENGTHS

Cells/Length M	Cells/Width N	Cap (pF)	Change (pF)	Change (%)
256	2	22.80688	—	—
256	4	22.52021	−0.287	−1.273%
256	8	22.37769	−0.143	−0.637%
256	16	22.30741	−0.070	−0.315%
256	32	22.27319	−0.034	−0.154%
256	64	22.25693	−0.016	−0.073%
256	128	22.24960	−0.007	−0.033%
256	256	22.24958	0.000	−0.000%
2	256	22.76381	—	—
4	256	22.47861	−0.285	−1.269%
8	256	22.34509	−0.134	−0.598%
16	256	22.28300	−0.062	−0.279%
32	256	22.25650	−0.027	−0.119%
64	256	22.24713	−0.009	−0.042%
128	256	22.24547	−0.002	−0.007%
256	256	22.24958	0.004	0.018%

analysis and using a coarse mesh in both dimensions (say, 4 × 4 cells), the total error can exceed 4%, an unacceptable situation for a number of applications, especially in filter design.

The parallel plate $(\varepsilon A/d)$ capacitance for this case is 221.136 pF. This means approximately 0.356 pF ± 0.01 pF is due to fringing capacitance and numerical error other than error due to cell width or cell length. Although our expectation is that most of this is fringing capacitance, which is discussed later; convergence analysis with respect to other error sources (e.g., numerical precision, fast Fourier transform (FFT) truncation, etc.) could be performed if desired.

The analyses were repeated for capacitors varying in plate separation (from 100 to 300 nM), change in dielectric constant (up to $\varepsilon_r = 1000$), and aspect ratio (from square to 2:1) with essentially the same results. Several cases are summarized in Tables II and III. Discrepancies for the low error region of Table III may be related to numerical precision (cell size is about 2 μM on a side).

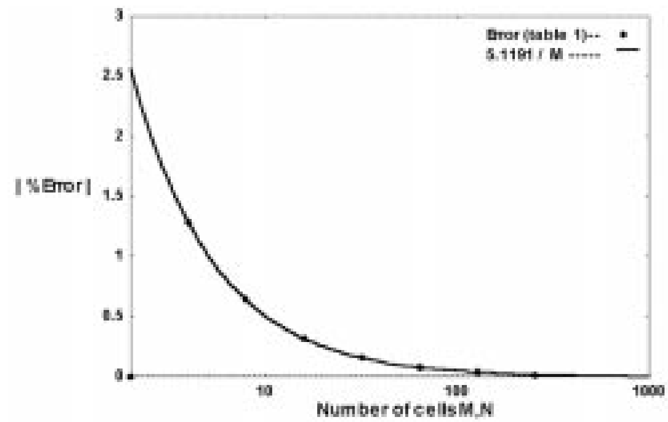

Fig. 5. The magnitude of the error in the capacitance calculation versus number of cells in the X- or Y-direction is shown in the dotted line. The error model $5.119/M$ is shown with the solid line.

Fig. 5 is a plot of the results from the % Change (error) column in Table I versus M or N showing the relationship between error in a given direction and the number of cells in that direction. As was stated earlier, the basis of Richardson extrapolation is to consider the result as a function of the discretization. Here, we modify this procedure slightly by considering the *error* to be a function of the discretization and model its behavior. Combining the x and y sources of error, a rational function of the form

$$\%E(M,N) = \frac{a_1}{M} + \frac{a_2}{N} \quad (1)$$

was fit to the data for both dimensions giving

$$\%E_{\text{sub}}(M,N) = \frac{5.1191}{M} + \frac{5.1191}{N} \quad (2)$$

where M, N are the number of cells in the x-, y-direction, respectively. After repeating this for the other tables (the ones shown here and several other cases) the coefficients a_1, a_2 in (1) were averaged resulting in the subsectioning error model

$$\%E_{\text{sub}}(M,N) = \frac{5.12}{M} + \frac{5.12}{N}. \quad (3)$$

In [2], it was found that error due to cell width can cancel error due to cell length in a transmission line. However, in this investigation, for capacitor subsectioning it was found that the errors always add. Thus, the error provided by the above model (3) may be simply subtracted from the calculated value, thus

$$C \approx C_{em} - \frac{\%E_{\text{sub}}(M,N)C_{em}}{100} \quad (4)$$

providing a more accurate result without the effort involved in a detailed convergence analysis (as in Tables I–III). For

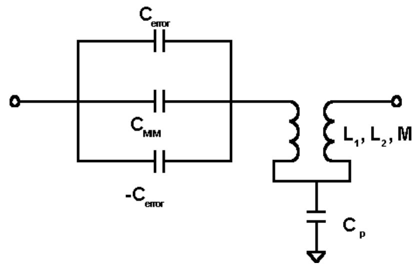

Fig. 6. Equivalent circuit model of a simulated series MIM capacitor including error compensation. $C_{\rm MIM} + C_{\rm error}$ is the calculated value of the series capacitance, $C_{\rm error}$ is the error in the calculation due to discretization given by (1). Adding $-C_{\rm error}$ removes this error.

the case of the series MIM capacitor, this results in a model which is compensated for errors due to discretization, as shown in Fig. 6. Alternatively, the size of the capacitor (either as analyzed or as fabricated) can be modified to compensate for analysis error. Of course, if extremely accurate results are needed, a convergence analysis should be used.

For the purposes of this paper, simple rectangular geometries have been studied, but there is no indication that this is a limitation of this type of analysis. For arbitrary shapes of capacitors, convergence of the error can still be tracked and extrapolated as was done for the rectangular capacitors.

III. Comparison With Analytical Techniques

The capacitance of a parallel-plate capacitor including the fringing capacitance can be calculated by modifying the method used in [6] and [7] where the parallel-plate capacitance is augmented by the capacitances due to the edges. This is done by considering the capacitor to be a degenerate transmission line and calculating the characteristic impedance and effective dielectric constant for the two types of lines formed by the edges of a MIM capacitor (using ***em*** or by some other means). The total capacitance for each transmission line can then be calculated. For this case, the two different types of transmission lines formed by the edges of the MIM capacitor (in Fig. 1) are parallel-plate for the x-dimension and microstrip for the y-dimension, yielding

$$C_w = \frac{\sqrt{\varepsilon_{{\rm eff}m}}}{cZ_{\rm omic}} w$$
$$C_1 = \frac{\sqrt{\varepsilon_{{\rm eff}p}}}{cZ_{\rm opp}} 1 \qquad (5)$$

for the y- and x-dimension total capacitances, respectively, where c is the phase velocity of light in vacuum, $Z_{\rm omic}, \varepsilon_{{\rm eff}m}$ are the characteristic impedance and the effective dielectric constant for the microstrip-like edges, and $Z_{\rm opp}, \varepsilon_{{\rm eff}p}$ are those for the parallel-plate edges of the capacitor.

The fringing capacitance per edge is then given by

$$C_{f{\rm mic}} = \frac{C_w - C_{\rm ppc}}{2}$$
$$C_{f{\rm pp}} = \frac{C_1 - C_{\rm ppc}}{2} \qquad (6)$$

for the microstrip and parallel-plate transmission-line parts of the capacitor where

$$C_{\rm ppc} = \frac{\varepsilon_r \varepsilon_o w l}{d} \qquad (7)$$

is the parallel-plate capacitance. Then the total capacitance (neglecting corner capacitance) is

$$C = C_{\rm ppc} + 2C_{f{\rm mic}} + 2C_{f{\rm pp}} \qquad (8)$$

which, when applied to the air dielectric case, gives a value of 22.191 pF and is in close agreement with the ***em*** calculated value including the error correction, which is 22.240 pF, leaving approximately 0.049 pF (∼0.22%) due to corner capacitance and error other than discretization error.

IV. Summary

As stated in [6], it is important to consider the cell size used in an electromagnetic simulator since computer time increases rapidly with the number of cells. Without quantitative knowledge of the error versus cell-size tradeoff, the designer does not know if a given cell size yields sufficient accuracy or if it is "overkill" resulting in a long simulation time. The authors believe there may be many more applications of Richardson extrapolation in the analysis of error in computational electromagnetics, as was also pointed out in [10].

The error model described in this paper can be used: 1) to select a discretization for a desired accuracy level and 2) to reduce the error for a given discretization by subtracting the error capacitance from the calculated value. The technique described allows a designer to achieve the desired level of simulation error while also realizing the minimum possible simulation time.

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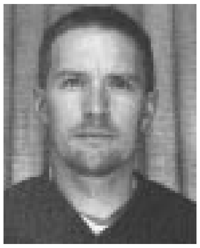

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