N-Port T-Networks and Topologically Symmetric Circuit Theory

James C. Rautio, *Fellow, IEEE*

***Abstract*—N-port Pi-networks can be extracted directly from Y-parameters. Likewise, two-port T-networks can be extracted directly from two-port Z-parameters. However, it appears that N-port T-networks have not been previously identified for $N > 2$, a strange topological asymmetry in circuit theory. We introduce the general N-port T-network and illustrate application by synthesis of an N-port T-network lumped model from Z-parameters for a three-port microstrip via to ground on GaAs. The existence of the N-port T-network partially relieves the noted circuit theory asymmetry; however, the T-network is structurally different from the Pi-network. We show that circuit theory becomes topologically perfectly symmetric under an appropriate change of variables when we consider g-parameters and derive circuit theory from the electromagnetically symmetric form of Maxwell's equations. The perfect topological symmetry requires both electric and magnetic (i.e., magnetic monopole) current. Practical application might be possible because effects identical to magnetic monopole current and charge have recently been experimentally observed and reported.**

***Index Terms*—Circuit theory, compact modeling, magnetic current, model synthesis, Pi-network, symmetry, tee-network, T-network, Y-parameters, Z-parameters.**

I. Background

RECIPROCAL Pi-network lumped models [see Fig. 1(a)] correspond to Y-parameters

$$\mathbf{Y} = \begin{bmatrix} y_{10} + y_{12} & -y_{12} \\ -y_{12} & y_{20} + y_{12} \end{bmatrix} \tag{1}$$

with generalization of N-port Y-parameters to N-port Pi-networks straightforward. Lumped reciprocal two-port T-networks [see Fig. 1(b)] correspond to two-port Z-parameters

$$\mathbf{Z} = \begin{bmatrix} z_{10} + z_{12} & z_{12} \\ z_{12} & z_{20} + z_{12} \end{bmatrix}. \tag{2}$$

A generalization for N-port Z-parameters and T-networks does not appear to have been previously identified. A limited three-port T-network is used to model a microstrip T-junction in [1]; however, that model cannot represent an arbitrary three-port Z-parameter matrix. A portion of a model for N-conductor transmission lines [2] is functionally and topologically similar to an N-port T-network; however, it is not identified as such.

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The author is with Sonnet Software Inc., North Syracuse, NY 13212 USA (e-mail: rautio@sonnetsoftware.com).

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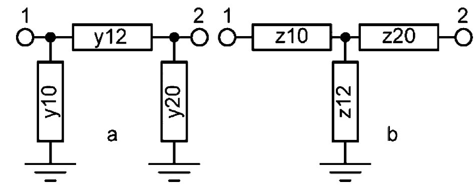

Fig. 1. (a) The Pi-network corresponds to Y-parameters and is easily generalized to N-ports. (b) The T-network corresponds to Z-parameters and generalization to N-ports appears to be previously unrecognized.

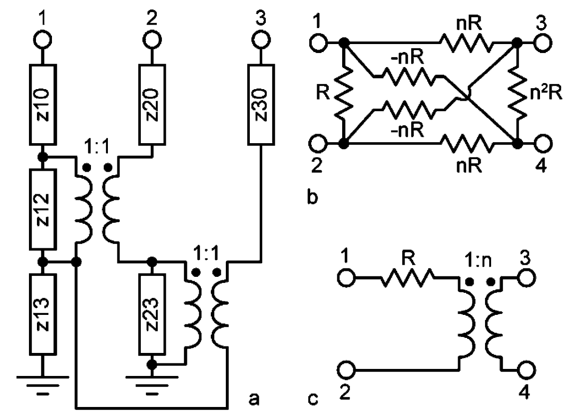

Fig. 2. (a) Three-port T-network corresponds to three-port Z-parameters and is easily generalized to N-ports. Other similar topologies are possible. (b) Resistive network approximates the ideal transformer in the limit as R goes to zero and is valid at all frequencies. The negative resistors are required. (c) Ideal transformer being modeled.

This study identifies the N-port T-network topology (Fig. 2) that corresponds to Z-parameters. Application of the T-network is illustrated with a synthesized lumped model for a multiport via to ground.

It is then noted that the T-network is topologically different from the Pi-network. This means circuit theory appears to have an inherent asymmetry in that neither the Pi-, nor the T-network can be derived from the other by a change of variables, by swapping current and voltage.

In contrast, the complete form of Maxwell's equations (which includes a fictional magnetic current) is perfectly symmetric in that a change of variables (i.e., swapping electric and magnetic field variables, etc.) yields the same set of equations. Circuit theory is a special case of Maxwell's equations. Thus, swapping current and voltage should result in the same topology for Pi- and T-networks. We resolve this contradiction by showing that when circuit theory also includes this fictional magnetic current, Pi- and T-networks also have this same symmetry. This author is unaware of any prior work combining both electric and magnetic current in circuit theory.

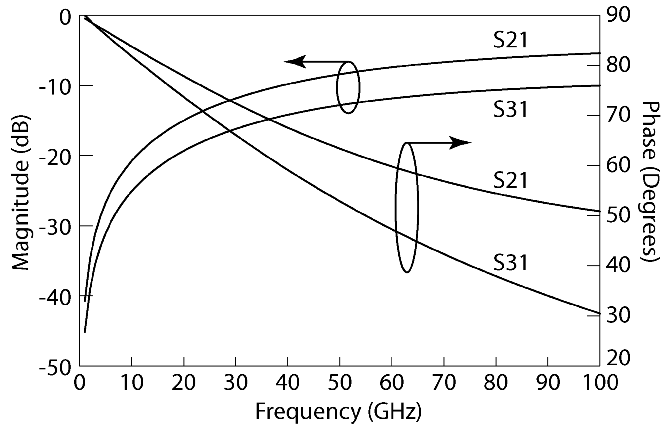

Fig. 3. Results for the T-network RLC lumped model for a three-port via to ground on GaAs (Fig. 4) synthesized from EM data are visually identical to the original EM data. Shown here are the isolation S-parameters.

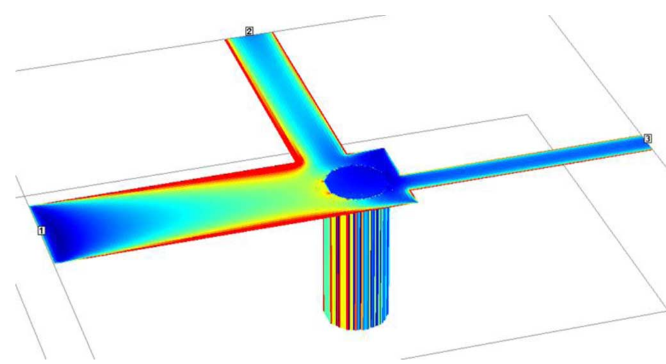

Fig. 4. Three-port microstrip via to ground on GaAs used in Fig. 3. Current distribution is shown with red (in online version) as high current and blue (in online version) as low current. Excitation is on port 1 (to the left).

II. N-Port T-Network Models

Fig. 2(a) shows a three-port model (not unique) corresponding to

$$\mathbf{Z} = \begin{bmatrix} z_{10}+z_{12}+z_{13} & z_{12} & z_{13} \\ z_{12} & z_{20}+z_{12}+z_{23} & z_{23} \\ z_{13} & z_{23} & z_{30}+z_{13}+z_{23} \end{bmatrix}. \quad (3)$$

Extending the three-port T-network of Fig. 2 to N-ports is straightforward requiring $N(N-1)/2-1$ ideal 1:1 transformers. We still refer to this topology as a T-network even though it no longer looks like a "T" because if any $N-2$ ports are terminated in open circuits, the remaining two ports form a T-network.

An exact transformer lumped model using only R, L, and C elements does not exist. Models are well known for two-port mutual inductors, e.g., [3] (T-model), in [4, Fig. 3] (Pi-model), and [5, Fig. 3(b)] (Pi-model). These models approximate ideal transformers in the limit as the inductance approaches zero. However, at low frequency and dc, the inductors in the model have low or zero reactance and the ideal transformer model fails due to numerical precision. At high frequency, the inductor reactances become large and the model fails because the inductor reactances must be small compared to the circuit impedances in which it is embedded. Typical applications of ideal transformers in modeling are presented in [1], [2], [4], [6], and [7].

An alternative model for an ideal four-port (each terminal is a port, all the ports have the same floating ground reference) transformer with series resistance, R, is shown in Fig. 2(b). This model is valid as long as R is small compared to the circuit impedances in which the transformer is embedded. Unlike a transformer model based on inductors, this model works at all frequencies, high and low, including dc. This model appears to be previously unreported. Note that the much more common two-port model for a transformer has the ground terminals for each port shorted together, contrary to the usual physical realization.

Note that for the special case of $n=1$, any two adjacent ports can be arbitrarily assigned the role of "primary" or "secondary" winding. Negative lumped elements are required for an RLC four-port transformer model regardless of whether the resistive or the inductive model is used. In both cases, the transformer model is absolutely stable and passive. A two-port transformer model is not suitable here because both ground terminals of such a model are at the same potential.

The object of generating an N-port T-network is to double the solution space of an existing lumped model synthesis [5]. In [5], we extract the admittance of the various branches of a Pi-network from frequency-domain electromagnetic (EM) analysis[1] generated Y-parameter data. A lumped model is then synthesized from the admittance data and an N-port Pi-network model is formed. The synthesis finds the best model within the solution space. Usually the model is physically significant. For example, a model of a capacitor looks like a capacitor. The solution space for a two-port is over a quarter billion possible topologies because each of the three branches of the Pi- or T-network may be any of over 600 different RLC topologies.

Full details of the synthesis are provided in [5] and are not repeated here. However, even with a solution space of over a quarter billion possible topologies, there are still structures for which an adequate Pi-network model cannot be found. We have found that, in this case, we can sometimes realize a successful synthesis by including N-port T-networks. N-port T-networks double the solution space (to over 600 million topologies) and allow topologies to be synthesized for previously intractable structures.

Fig. 3 shows results for a synthesized T-network model of a three-port microstrip via to ground on 100-μm GaAs (Fig. 4). EM calculated and modeled results are visually identical for all nine complex S-parameters. The average signal to error ratio (SER) (the magnitude of the vector difference between modeled and calculated compared to the calculated magnitude) is over 60 dB. The best Pi-network model SER is 20 dB less.

Whether a Pi- or a T-network is superior (by whatever criteria are important to the designer) depends on the circuit being modeled and any specific design objectives. On occasion, it is possible to produce a better T-network model by inserting both a positive and an identical negative branch model and then combining two or more of the series impedances and synthesizing a branch model for the combined impedance. This was done

[1] [Online]. Available: http://www.sonnetsoftware.com

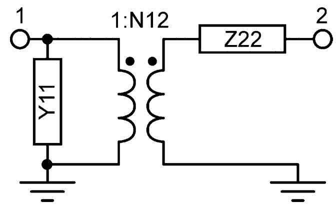

Fig. 5. Two-port hybrid Pi–T-network corresponding to (4). Easily generalized to N-ports, Y_{11} is a multiport Pi-network, Z_{22} is a multiport T-network, and N_{12} is an array of transformers with, in general, a complex transformation ratio.

for several of the branches in the three-port via model. For example, the first branch of the port 1 series connection is actually $z_{10}+z_{13}$, and an additional $-z_{13}$ is inserted into the series connection. The synthesized model uses 69 lumped elements.

In [2, Fig. 3(a)], a topology similar to the N-port T-network is illustrated and used to model the series per unit length $R+j\omega L$ of a multiconductor transmission line. While similar to the T-network described here ([5] requires $N+1$ additional transformers), it is not identified as an N-port T-network.

Some of the transmission line models described in [2] (e.g., [2, Fig. 7]) can be synthesized using the techniques described in this paper. Specifically, the $R+j\omega L$ portion is realized by synthesizing an N-port T-network for an N-conductor shorted stub. The $G+j\omega C$ portion can be synthesized by synthesizing an N-port Pi-network model [5] for an identical N-conductor open-circuited stub. $R, L, G,$ and C are, in general, a function of frequency. The synthesized models of [5] are composed of lumped elements that are independent of frequency. A perfect open circuit is realized by terminating the stub in a perfect magnetic conducting wall. A perfect short circuit uses a perfect electric conducting wall. The two models are then combined, as illustrated in [2]. Using this approach, more general frequency-dependent branch models are possible; however, passivity and stability are not guaranteed without additional effort. Additional N-conductor line models can be synthesized by using the technique of [5] to synthesize a $2N$-port Pi-network or the technique of this paper to synthesize a $2N$-port T-network.

III. Hybrid Networks

In the same fashion we use T-networks above to enlarge the lumped model synthesis space, we can also use hybrid networks, networks that are a combination of Pi- and T-networks. In this case, we divide the N-ports into two groups. The first group of ports is terminated with short-circuit terminations and the second group is terminated with open-circuit terminations. The corresponding g-parameter [3] system of equations for the case of a two-port is

$$\begin{pmatrix} I_1 \\ V_2 \end{pmatrix} = \begin{bmatrix} Y_{11} & -n_{12} \\ n_{12} & Z_{22} \end{bmatrix} \begin{pmatrix} V_1 \\ I_2 \end{pmatrix}. \tag{4}$$

The hybrid matrix corresponds to a reciprocal system if it is antisymmetric as shown. The corresponding model is shown in Fig. 5. For the N-port case, each current and voltage of (4) becomes a vertical vector and the elements of the matrix themselves become matrices. The Y_{11} model of Fig. 5 becomes a multiport Pi-network, the Z_{22} model becomes a multiport T-network. The total number of ports is the sum of the Y_{11} and Z_{22} ports. The transformer becomes an array of transformers with primaries connecting in parallel with the Pi-network ports and secondaries connecting in series (totem-pole style) with the T-network ports. In general, the transformation ratio n is complex except at dc. Given a g-parameter matrix (4), one can convert any port or ports from Y to Z, or from Z to Y by performing Gaussian elimination type of operations.

IV. Fully Symmetric Circuit Theory

Section II illustrates practical application of N-port T-networks. However, it does not remove the fundamental topological asymmetry in circuit theory. While we now have circuit N-port topologies corresponding to Y-parameters (Pi-networks) and Z-parameters (T-networks), the two topologies are different. This is a fundamental theoretical quandary.

This topological asymmetry is completely resolved if we invoke the mathematically symmetric form of Maxwell's equations, specifically, the form that includes a fictitious (i.e., not seen in nature) flow of magnetic monopoles, the magnetic current. In the time–harmonic form [8], we have

$$\nabla \times \mathbf{E} = -j\omega\mu\mathbf{H} - \mathbf{M} \tag{5}$$

$$\nabla \times \mathbf{H} = +j\omega\varepsilon\mathbf{E} + \mathbf{J} \tag{6}$$

where $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, ω is radian frequency, μ is permeability, ε is permittivity, $\mathbf{M}$ is the magnetic current, and $\mathbf{J}$ is the electric current. In a given material, the flow of electric conduction current is proportional to the $\mathbf{E}$-field. Symmetrically, the flow of magnetic conduction current is proportional to the $\mathbf{H}$-field. The constants of proportionality are the electric and magnetic conductivity, which form the constitutive relationships

$$\mathbf{J} = \sigma_e\mathbf{E} \tag{7}$$

$$\mathbf{M} = \sigma_m\mathbf{H} \tag{8}$$

where (7) is the field equivalent of Ohm's law. When magnetic conduction current is allowed, the intrinsic impedance of a medium is

$$\eta = \sqrt{\frac{j\omega\mu + \sigma_m}{j\omega\varepsilon + \sigma_e}} \tag{9}$$

where, for example, $\eta = 377\ \Omega$ for free space. Maxwell's equations have solutions for problems that include $\sigma_e = \infty$ [a perfect electric conductor (PEC)] or that include $\sigma_m = \infty$ [a perfect magnetic conductor (PMC)]. Within the framework of Maxwell's equations, a perfect conductor can be either PEC or PMC, but it cannot be both because η becomes indeterminate. In addition, a material that is a PMC always has zero tangential $\mathbf{H}$-field. This means that electric current cannot flow in it and it is necessarily an open circuit to electric current. Likewise, a PEC material always has zero tangential $\mathbf{E}$-field, and thus, is an open circuit for magnetic current.

To map the symmetric Maxwell's equations into symmetric circuit theory, we need four circuit theory variables

$$V = \int \mathbf{E} \cdot \mathbf{dl} \tag{10}$$

$$I = \oint \mathbf{H} \cdot \mathbf{dl} = \oint \mathbf{J} \cdot \mathbf{ds} \tag{11}$$

$$U = \int \mathbf{H} \cdot \mathbf{dl} \tag{12}$$

$$K = \oint \mathbf{E} \cdot \mathbf{dl} = \oint \mathbf{M} \cdot \mathbf{ds} \tag{13}$$

where V and I are the electric voltage and current, U and K are the magnetic voltage and current, and $\mathbf{dl}$ and $\mathbf{ds}$ are infinitesimal lengths and areas. The paths and areas for the magnetic current and voltage integrals are exactly the same as for the electric currents and voltages.

To illustrate the effect of including the magnetic conduction current, we solve a classic EM problem, current flowing at dc (i.e., $\omega = 0$) in a circular cylinder (a wire) of conductivity σ_e. With no magnetic conduction current, $\sigma_m = 0$, the electric and magnetic fields inside the cylinder are

$$\mathbf{E} = E_0 \mathbf{u_z} \tag{14}$$

$$\mathbf{H} = \frac{E_0 \sigma_e r}{2} \mathbf{u}_\phi \tag{15}$$

where the conductor lies along the z-axis with radial coordinate r and azimuthal coordinate ϕ and $\mathbf{u}$ are unit vectors.

When we allow a finite nonzero magnetic conductivity, the solution to Maxwell's equations becomes

$$\mathbf{E} = E_0 I_0(\sqrt{\sigma_e \sigma_m} r) \mathbf{u_z} \tag{16}$$

$$\mathbf{H} = E_0 \sqrt{\frac{\sigma_e}{\sigma_m}} I_1(\sqrt{\sigma_e \sigma_m} r) \mathbf{u}_\phi \tag{17}$$

where $I_N(x)$ is the modified Bessel function of the first kind of order N. E_0 is the electric field in the center of the conductor. Electric and magnetic currents are found by application of (7) and (8). Notice that a z-directed electric current induces a solenoidal ϕ-directed magnetic current. The magnetic field is zero in the center. Both the electric and magnetic field gradually increase with r approaching the outer edge. In the limit, as σ_m approaches zero, (16) and (17) converge to (14) and (15).

This solution assumes an electric current source placed on the ends of the wire generating E_0. Due to the symmetry of Maxwell's equations, we also find a solution, identical in form to that above, corresponding to a magnetic current source generating a z-directed H_0 by a change of variables.

We can use electric current to generate magnetic voltage by splitting the cylinder in two along a plane containing the center (the z-axis) of the cylinder. This interrupts the azimuthal magnetic current forming a magnetic voltage across the gap between the two half cylinders.

Thus, as illustrated by the above three cases, a symmetric circuit theory model must allow for electric current generating electric voltage, magnetic current generating magnetic voltage, electric current generating magnetic voltage, etc.

Loss in the conductor is [8]

$$P_L = \iiint (\mathbf{E} \cdot \mathbf{J}^* + \mathbf{H}^* \cdot \mathbf{M}) d\tau \tag{18}$$

with the integral taken over the volume of the conductor and the asterisk indicating complex conjugate. Application of this integral to the fields for the combined electric and magnetic conductor (16) and (17) illustrate that the loss (and thus, electrical resistance) is increased by magnetic conductivity even though there is no magnetic source. If we apply (18) to the solution assuming a magnetic current source, we similarly find that the total loss is modified by the electric conductivity even though we have no electric source. Thus, the electrical aspects of a symmetric circuit theory model must be modified by magnetic conductivity, and the magnetic aspects must be modified by electrical conductivity.

We now have sufficient information to form a topologically symmetric circuit theory model. First, in normal circuit theory, there are two variables associated with each port, electric voltage (10) and electric current (11). With magnetic current included, there are two additional variables, magnetic voltage (12) and magnetic current (13) for each physical port. We have both electric and magnetic voltage and current. To determine Y-parameters, we terminate all N physical ports in electric short circuits (PEC).

Next, we represent each of the physical ports with two ports in the circuit theory model, one to represent electric current and voltage, the other to represent magnetic current and voltage. Note that in using a PEC termination, we effectively short circuit the electric current ports and simultaneously open circuit the magnetic current ports. Remember, a PEC short circuit must simultaneously be a PMC open circuit if we are to remain within the framework of Maxwell's equations. Evaluations of all electric and magnetic currents and voltages under these terminating conditions with appropriately applied magnetic and electric sources yields all needed information for a $2N$-port g-parameter matrix (4) and its associated Pi–T model (Fig. 5). Note carefully that for this combined electric and magnetic current circuit model, all port voltages and currents on the right-hand side of Fig. 5 are magnetic voltages and currents. If we set all magnetic currents and voltages to zero, the model reduces to an electric current Pi-network.

Alternatively, we can use PMC port terminations and all electric current ports are terminated in open circuits and all magnetic current ports are terminated in short circuits. Thus, the model of Fig. 5 combined with (4) yields a topologically identical model with electric current ports changed to magnetic current ports, and magnetic current ports changed to electric current ports. If we now set all magnetic currents and voltages to zero, we have an electric current T-network.

Thus, a general circuit theory, incorporating both electric and magnetic current, is topologically symmetric. Regardless of whether we use PEC or PMC terminations to characterize an N-port structure, the topology of the corresponding circuit is identical. The topological asymmetry we see in electric current circuit theory is because we live in a universe that favors electric monopoles. A circuit theory including both electric and magnetic current, as described in this section, is a peek into what circuit theory would look like if we were to live in a universe that gave equal stature to both electric and magnetic monopoles. If magnetic monopoles were favored, circuit theory would be exactly the same with only a change of variables.

It was initially thought that this fully symmetric circuit theory would be a curiosity of no practical use. However, recently effects have been experimentally observed that are identical to the effects of magnetic charge and current [9]. Total magnetic charge is observed to be zero and it is emphasized that these observations cannot be deemed a discovery of magnetic monopoles. In addition, the magnetic current and charge are presently observed in materials that are electrical insulators. However, future work [10] will investigate magnetic current and charge in metals, which allow electric current to flow as well. If these metals do indeed also support magnetic current (which would, in turn, modify the electric current, as described above), and Maxwell's equations apply, then the perfectly symmetric circuit theory reported here can be applied to model resulting structures.

V. Conclusion

For what is possibly the first time, we describe the general N-port T-network and demonstrate its utility in synthesizing a T-network model for a three-port microstrip via to ground. In doing so, we also introduce, again possibly for the first time, a $1{:}n$ transformer model based on resistors, which is valid to dc. We also discuss g-parameters and their associated hybrid Pi–T-network. We then investigate the symmetric form of Maxwell's equations (i.e., with fictional magnetic current included) and show that this leads to a topologically symmetric circuit theory, where Y- and Z-parameters (i.e., parameters evaluated using short/PEC or open/PMC circuit port terminations) each correspond to the same topology, demonstrating that a general circuit theory is, in fact, topologically symmetrical. This symmetry is present only if both electric and magnetic current are allowed, a situation which might soon actually be observed experimentally.

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James C. Rautio (S'77–M'78–SM'91–F'00) received the B.S.E.E. degree from Cornell University, Ithaca, NY, in 1978, the M.S. degree in systems engineering from the University of Pennsylvania, Philadelphia, in 1982, and the Ph.D. degree in electrical engineering from Syracuse University, Syracuse, NY, in 1986.

From 1978 to 1986, he was with General Electric, initially with the Valley Forge Space Division, and then with the Syracuse Electronics Laboratory. During this time, he developed microwave design and measurement software and designed microwave circuits on alumina and on GaAs. From 1986 to 1988, he was a Visiting Professor with Syracuse University and Cornell University. In 1988, he joined Sonnet Software Inc., Liverpool, NY, full time, a company he founded in 1983. In 1995, Sonnet Software Inc. was listed on the Inc. 500 list of the fastest growing privately held U.S. companies, the first microwave software company ever to be so listed. Today, Sonnet Software Inc. is the leading vendor of high-accuracy 3-D planar high-frequency EM analysis software.

Dr. Rautio was an elected IEEE Microwave Theory and Techniques Society (IEEE MTT-S) Distinguished Microwave Lecturer (2005–2007), lecturing on the life of James Clerk Maxwell. He was the recipient of the 2001 IEEE Microwave Theory and Techniques Society (IEEE MTT-S) Microwave Application Award.